CHAPTER 3

LINEAR DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

1. LINEAR DIFFERENTIAL EQUATIONS (L.D.E.) ►

In this chapter we discuss the non homogeneous linear differential equation of first order which takes the form

$$\frac{dy}{dx} + P(x)y = Q(x) \dots\dots\dots\dots (1)$$

And then we will investigate how to solve nonlinear D.E. can be transformed into L.D.E.

■ NOTE THAT►

(1) The D.E. in the form $\frac{dy}{dx} + P(x)y = 0$ is called homogeneous linear differential equation of first order.

(2) The D.E. may be linear in the variable x, such as

$$\frac{dx}{dy} + P(y)x = Q(y)$$

THE GENERAL SOLUTION OF L.D.E.►

The general solution of the differential equation (1) is given by

$$I(x)\, y = \int I(x)\, Q(x)dx + c,$$

where $I(x) = e^{\int P(x)dx}$ is the integrating factor of equation (1).

EXAMPLE 1.1►

Solve the D.E.

$$y' + \frac{3}{x}y = \frac{e^x}{x^3} + 4$$

SOLUTION

Since the D.E. is linear then $P(x) = \frac{3}{x}$, $Q(x) = \frac{e^x}{x^3} + 4$.

Its integrating factor is

$$I(x) = e^{\int P(x)dx} = e^{\int \frac{3}{x}dx} = e^{3lnx} = x^3$$

So, the general solution can be written as

$$yx^3 = \int x^3 \left(\frac{e^x}{x^3} + 4\right) dx + c$$

$$yx^3 = e^x + x^4 + c$$

$$y = \frac{e^x}{x^3} + x + \frac{c}{x^3}$$

EXAMPLE 1.2►

Find the general solution of the D.E.

$$\frac{dy}{dx} + \frac{1}{x}y = e^x$$

SOLUTION

The D.E. is linear, with $P(x) = \frac{1}{x}$, $Q(x) = e^x$.

Its integrating factor is

$$I(x) = e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{lnx} = x$$

So, we can obtain the general solution from

$$I(x)\, y = \int I(x)\, Q(x)dx + c$$

$$xy = \int x\, e^x dx + c$$

Using integral by parts the solution is

$$xy = e^x(x-1) + c$$

EXAMPLE 1.3►

Find the general solution of the D.E.

$$\cos x \frac{dy}{dx} - y \sin x = \sin x \cos x$$

SOLUTION

Divided the equation by $\cos x$, we get

$$y' - \mathrm{y} \tan x = \sin x$$

The D.E. is linear, with $P(x) = -\tan x$, $Q(x) = \sin x$.

Integrating factor is

$$I(x) = e^{\int P(x)dx} = e^{-\int \tan x dx} = e^{\ln \cos x} = \cos x$$

So, the general solution is

$$y \cos x = \int \cos x \sin x \, dx + c = \frac{1}{2}\int \sin 2x \, dx + c$$

$$y \cos x = \frac{-1}{4}\cos 2x + c$$

EXAMPLE 1.4►

Find the general solution of the D.E.

$$y' + \mathrm{y} \tan x = sec\, x$$

SOLUTION

The D.E. is linear, with $P(x) = \tan x$, $Q(x) = \sec x$.

Integrating factor is

$$I(x) = e^{\int P(x)dx} = e^{\int \tan x dx} = e^{\ln \sec x} = \sec x$$

So, the general solution is

$$y \sec x = \int \sec^2 x \, dx + c$$

$$y \sec x = \tan x + c$$

EXAMPLE 1.5►

Find the general solution of the D.E.

$$y = x(y' - x \cos x)$$

SOLUTION

The equation can be written in the form

$$y' - \frac{1}{x} y = x \cos x$$

The D.E. is linear, with $P(x) = -\frac{1}{x}$, $Q(x) = x \cos x$.

Integrating factor is

$$I(x) = e^{\int P(x)dx} = e^{-\int \frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}$$

So, the general solution is

$$y \frac{1}{x} = \int \cos x \, dx + c$$

$$y \frac{1}{x} = \sin x + c$$

$$y = \mathrm{x} \sin x + cx$$

EXAMPLE 1.6►

Find the general solution of the D.E.

$$y' - \frac{2}{x+1} y = (x+1)^3$$

SOLUTION

The D.E. is linear, with $P(x) = -\frac{2}{x+1}$, $Q(x) = (x+1)^3$.

Integrating factor is

$$I(x) = e^{\int P(x)dx} = e^{-\int \frac{2}{x+1}dx} = \frac{1}{(x+1)^2}$$

So, the general solution is

$$y\frac{1}{(x+1)^2} = \int \frac{(x+1)^3}{(x+1)^2}dx + c = \frac{1}{2}(x+1)^2 + c$$

$$\therefore y = \frac{1}{2}(x+1)^4 + c(x+1)^2$$

EXAMPLE 1.7►

Find the general solution of the D.E.

$$y' - \frac{n}{x}y = x^n e^x$$

SOLUTION

The D.E. is linear, with $P(x) = -\frac{n}{x}$, $Q(x) = x^n e^x$.

Its integrating factor is

$$I(x) = e^{\int P(x)dx} = e^{-\int \frac{n}{x}dx} = \frac{1}{x^n}$$

So, the general solution is

$$y\frac{1}{x^n} = \int \frac{x^n e^x}{x^n}dx + c = e^x + c$$

$$\therefore y = x^n(e^x + c)$$

EXAMPLE 1.8►

Find the general solution of the D.E.

$$(x-1)y' - y = 2(x-1)^3$$

SOLUTION

The equation can be written in the linear form

$$y' - \frac{1}{x-1}y = 2(x-1)^2,$$

where $P(x) = -\frac{1}{x-1}$, $Q(x) = 2(x-1)^2$.

Integrating factor is

$$I(x) = e^{\int P(x)dx} = e^{-\int \frac{1}{x-1}dx} = e^{-\ln(x-1)} = \frac{1}{x-1}$$

So, the general solution is

$$\frac{y}{x-1} = 2\int (x-1)dx + c$$

$$\frac{y}{x-1} = (x-1)^2 + c$$

$$y = (x-1)^3 + c(x-1)$$

2. BERNOULLI'S EQUATION ►

The differential equation in the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \ldots\ldots\ldots (2)$$

is called Bernoulli's differential equation, it can be transformed into a L.D.E. as follows:

(i) Divided equation (2) by y^n

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$$

(ii) Using the substitution

$$z = y^{1-n}$$

$$\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

Eq. (2) will be transformed to

$$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x)$$

which is L.D.E.

EXAMPLE 2.1►

Find the general solution of the D.E.

$$y' + 2y = e^x y^2$$

SOLUTION

Divided both sides by y^2, then

$$y^{-2}y' + 2y^{-1} = e^x$$

Let $z = y^{-1} \rightarrow \frac{dz}{dx} = -y^{-2}\frac{dy}{dx}$

Substitute in the D.E. we get

$$\frac{dz}{dx} - 2z = -e^x$$

This is linear D.E. , with $P(x) = -2$, $Q(x) = -e^x$.

Integrating factor is

$$I(x) = e^{-\int 2dx} = e^{-2x}$$

The solution will take the form

$$z\, e^{-2x} = -\int e^x e^{-2x} dx + c$$

$$z\, e^{-2x} = e^{-x} + c$$

Substitute by $z = y^{-1}$, the general solution become

$$y(\, e^x + c\, e^{2x}) = 1$$

EXAMPLE 2.2►

Find the general solution of the D.E.

$$xy' + y = xy^2 \ln x$$

SOLUTION

Divided both sides by xy^2, then

$$y^{-2}y' + \frac{1}{x}y^{-1} = \ln x$$

Let $z = y^{-1} \rightarrow \frac{dz}{dx} = -y^{-2}\frac{dy}{dx}$

Substitute in the D.E. we get

$$\frac{dz}{dx} - \frac{1}{x}z = -\ln x$$

This is linear D.E., with $P(x) = -\frac{1}{x}$, $Q(x) = -\ln x$.

Integrating factor is

$$I(x) = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

The solution will take the form

$$\frac{z}{x} = -\int \frac{1}{x}\ln x\, dx + c$$

$$\frac{z}{x} = \frac{-(\ln x)^2}{2} + c$$

Substitute by $z = y^{-1}$, the general solution become

$$\frac{1}{y} = \frac{-x(\ln x)^2}{2} + cx$$

EXAMPLE 2.3►

Find the general solution of the D.E.

$$\frac{dy}{dx} - y = xy^5$$

SOLUTION

Divided both sides by y^5, then

$$y^{-5}y' - y^{-4} = x$$

Let $z = y^{-4} \rightarrow \frac{dz}{dx} = -4y^{-5}\frac{dy}{dx}$

Substitute in the D.E. we get

$$\frac{dz}{dx} + 4z = -4x$$

This is linear D.E., its integrating factor is

$$I(x) = e^{\int 4dx} = e^{4x}$$

The solution will take the form

$$e^{4x}z = -\int 4xe^{4x}dx + c$$

$$e^{4x}z = -xe^{4x} + \frac{1}{4}e^{4x} + c$$

But $z = y^{-1}$, then the general solution become

$$\frac{1}{y^4} = \frac{1}{4} - x + ce^{-4x}$$

EXAMPLE 2.4►

Find the general solution of the D.E.

$$y' + \frac{1}{x}y = (x^2 + 5x)y^3$$

SOLUTION

Divided both sides by y^3, then

$$y^{-3}y' + \frac{1}{x}y^{-2} = (x^2 + 5x)$$

Let $z = y^{-2} \rightarrow \frac{dz}{dx} = -2y^{-3}\frac{dy}{dx}$

Substitute in the D.E. we get

$$\frac{dz}{dx} - \frac{2}{x}z = -2(x^2 + 5x)$$

This is linear D.E., with $P(x) = -\frac{2}{x}$, $Q(x) = -2(x^2 + 5x)$.

Integrating factor is

$$I(x) = e^{-\int \frac{2}{x} dx} = e^{-2\ln x} = \frac{1}{x^2}$$

The solution will take the form

$$\frac{z}{x^2} = -2\int \frac{1}{x^2}(x^2 + 5x)dx + c$$

$$\frac{z}{x^2} = -2x - 10\ln x + c$$

Substitute by $z = y^{-2}$, the general solution become

$$\frac{1}{y^2} = -2x^2(x + 5\ln x) + cx^2$$

EXAMPLE 2.5►

Find the general solution of the D.E.

$$y' - \left(1 + \frac{1}{x}\right)y = -2e^x y^2$$

SOLUTION

Divided both sides by y^2, then

$$y^{-2}y' - \left(1 + \frac{1}{x}\right)y^{-1} = -2e^x$$

Let $z = y^{-1} \rightarrow \frac{dz}{dx} = -y^{-2}\frac{dy}{dx}$

Substitute in the D.E. we get

$$\frac{dz}{dx} + \left(1 + \frac{1}{x}\right)z = 2e^x$$

This is linear D.E., with $P(x) = \left(1 + \frac{1}{x}\right)$, $Q(x) = 2e^x$

Integrating factor is

$$I(x) = e^{\int \left(1 + \frac{1}{x}\right)dx} = e^{x + \ln x} = xe^x$$

The solution will take the form

$$xe^x z = 2\int xe^{2x}dx + c$$

By integral and substitute $z = y^{-1}$, the general solution become

$$\frac{xe^x}{y} = \left(x - \frac{1}{2}\right)e^{2x} + c$$

EXERCISES 1

Find the general solution of the following D.E.

(1) $y' + y = e^{-x}$

(2) $\frac{dy}{dx} + y = 1 + x$

(3) $\frac{dy}{dx} - \frac{2}{x+1}y = (x+1)^3$

(4) $x\frac{dy}{dx} - 4y = x^3$

(5) $y' + y\cot x = cosec\ x$

(6) $y'\cos x + y\sin x = 1$

(7) $y' - \frac{2}{x}y = \left(1 - \frac{2}{x}\right)e^x$

(8) $(1+x^2)y' + xy = 1$

(9) $(2x+1)y' - 2y = 4x$

(10) $x^2y' + xy = -1$

(11) $\frac{dy}{dx} + \frac{1}{x}y = \frac{1}{x^2}y^2$

(12) $y' - 4xy = x^3y^2$

(13) $y' - \frac{1}{x}y = x^3y^2$

(14) $y\frac{dy}{dx} + \frac{1}{2}y^2 = \sin x$

(15) $y' + y\cot x = y^{-1}cosec^2 x$

(16) $xy' + y = y^2\ln x$

(17) $\frac{dy}{dx} + \frac{y+3}{x} = x^2(y+3)^3$

(18) $xy' + 2y = -x^5e^xy^3$

(19) $2xyy' + (1+x)y^2 = e^x$

(20) $y' + y = xy^3$